VANISHING SLIDING MOTIONS OF MECHANICAL SYSTEMS WITH DRY FRICTION

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The energy-dissipating motions of frictional surfaces that involve some degree of sliding are considered, as well as motions without sliding (relative rest or pure rolling) in which energy may be conserved. It is natural to assume that if the energy influx into the system is small, the initial motions with sliding become motions without sliding. We know, for example, that a heavy homogeneous disc rolling and sliding along a rough horizontal straight line begins to roll without sliding after a finite time; the time $t - t_0$ of the transient process can be made arbitrarily small if v_0 (the initial sliding velocity) is sufficiently small. Do the same phenomena occur with general frictional systems? This paper presents an investigation of the sufficient conditions under which such phenomena do indeed occur.

1. Let us begin with an example and examine the case of a heavy nonhomogeneous disc of radius ρ rolling (and generally sliding) along the rough horizontal x-axis. We direct the stationary y-axis vertically upward and construct the radius r from the center of the disc o to its center of gravity G. Let φ be the angle between this radius and the y-axis, x the abcissa of the center of the disc, m its mass, j^2m the central moment of inertia, v the velocity of the point P of the disc in contact with the x-axis, N the normal reaction, and R the tangential reaction at the point P. If the velocity $v = x^{\cdot} - \varphi^{\cdot}\rho \neq 0$, the force R is equal to kN in magnitude and is directed opposite to the velocity v, i.e. $R = -kN$ sign v, where k = const is the coefficient of friction. The normal reaction $N > 0$; this means that it can only be directed upward. Denoting the kinetic energy by T, the force function by U, and the virtual work of the tangential force by δA, we obtain

$$2T = m\,[x^{\cdot 2} + 2x^{\cdot}\varphi^{\cdot} r\cos\varphi + (r^2 + j^2)\,\varphi^{\cdot 2}], \qquad U = -\,mgr\cos\varphi$$
$$\delta A = -\,kN \operatorname{sign} v\,(\delta x - \rho\delta\varphi)$$

We construct the equations of motion

$$m\,(x^{\cdot\cdot} + \varphi^{\cdot\cdot} r\cos\varphi - \varphi^{\cdot 2} r\sin\varphi) = -\,kN \operatorname{sign} v, \tag{1.1}$$
$$m\,(x^{\cdot\cdot} r\cos\varphi + \varphi^{\cdot\cdot} r^2 + \varphi^{\cdot\cdot} j^2) = kN\rho \operatorname{sign} v + mgr\sin\varphi$$

To determine N we make use of the theorem on the motion of the center of mass along the y-axis,

$$m\,(\varphi^{\cdot\cdot} r \sin\varphi + \varphi^{\cdot 2} r \cos\varphi) = mg - N \tag{1.2}$$

Multiplying the first equation of system (2.1) by $-r\cos\varphi$, and adding the second equation to the result, we have

$$[1 \pm \alpha k\,(\rho + r\cos\varphi)]\,N = (g + \varphi^{\cdot 2} r \cos\varphi)\,\frac{mj^2}{r^2 \sin^2\varphi + j^2}$$
$$\alpha = r\sin\varphi\,(r^2\sin^2\varphi + j^2)^{-1} \tag{1.3}$$

It is clear from this that N is independent of $x^{\cdot}$, and therefore of the absolute value of v, and depends solely on sign v (we have written $\pm$ instead of sign v in the expression in square brackets). It is also a simple matter to point out initial conditions and a mass distribution such that N is negative. In this case we encounter Painlevé's paradox [1], and the initial hypotheses turn out to be insufficient to determine the motion. But if the latter case is not encountered, then, substituting N from equation (1.3) in equations (1.1), we obtain the equations of motion. The motion will proceed according to these equations until such time as $v = x^{\cdot} - \varphi^{\cdot}\rho$ becomes 0.

Let us take this as the initial instant $t = 0$ and assume that what follows is pure rolling. In this case the system loses one degree of freedom and is subject to the integrable nonholonomic constraint $x^{\cdot} = \varphi^{\cdot}\rho$. The equation of motion becomes

$$\varphi^{\cdot\cdot}\,(r^2 + \rho^2 + j^2 + 2\rho r\cos\varphi) - \varphi^{\cdot 2}\rho r\sin\varphi = gr\sin\varphi \tag{1.4}$$

The theorem on the motion of the center of mass yields equations for determining R and N_1 (the tangential and normal reactions for pure rolling),

$$\varphi^{\cdot\cdot}\,(\rho + r\cos\varphi) - \varphi^{\cdot 2} r \sin\varphi = R\,/\,m$$
$$g + r^2\varphi^{\cdot 2}\cos\varphi + \varphi^{\cdot\cdot} r\sin\varphi = N_1\,/\,m$$

Solving them, we have

$$R\,/\,m = \frac{r\sin\varphi\,(g + \varphi^{\cdot 2}\rho)\,(\rho + r\cos\varphi)}{r^2 + j^2 + \rho^2 + 2\rho r\cos\varphi} - \varphi^{\cdot 2} r\sin\varphi$$
$$N_1\,/\,m = \frac{r^2\sin^2\varphi\,(g + \varphi^{\cdot 2}\rho)}{r^2 + j^2 + \rho^2 + 2\rho r\cos\varphi} - r\varphi^{\cdot 2} + g \tag{1.5}$$

If the inequality $|\,R\,| < kN_1$ is satisfied at the initial instant for $\varphi = \varphi_0$, $\varphi^{\cdot} = \varphi_0^{\cdot}$ the assumption as regards rolling is valid, and motion occurs in accordance with equation (1.4) until such time as the inequality $|\,R\,| < kN_1$ is violated.

It is natural to expect that any close initial conditions $\varphi_0 + \Delta\varphi$, $\varphi_0^{\cdot} + \Delta\varphi^{\cdot}$, $v_0 \neq 0$ corresponding to small $|\,\Delta\varphi\,|$, $|\,\Delta\varphi^{\cdot}\,|$, $|\,v_0\,|$, lead to motion in which sliding soon vanishes and rolling continues for some time thereafter (possibly for an infinite period).

As is shown below, such is always the case if the inequality $|\,R\,| < kN$ is satisfied in addition to $|\,R\,| < kN_1$ at the initial instant; the N appearing in the former inequality is taken from formula (2.3) for small values of $|\,\Delta\varphi\,|$, $|\,\Delta\varphi_0\,|$, $|\,v_0\,|$.

We denote the limiting value of N as $\Delta\varphi \to 0$, $\Delta\varphi^{\cdot} \to 0$, $v_0 \to 0 + 0$ by N_2, the limiting value of N as $\Delta\varphi \to 0$, $\Delta\varphi^{\cdot} \to 0$, $v_0 \to 0 - 0$, by N_3, and lim N_1 as $\Delta\varphi \to 0$, $\Delta\varphi_0^{\cdot} \to 0$.

by N_0. Formulas (1.3) and (1.5) indicate that they may all be distinct. In addition, N_2 or N_3 may turn out to be negative for positive N_0. This situation leads to a paradox. In the more complicated cases it may be that the equations similar to (1.3) and (1.5) have several solutions. Since analysis of such general equations is quite a complex matter, we impose limitations 2.1,....2.4, on the properties of their solutions and are then able to proceed with our proof.

Such problems invariably arise in the investigation of the motion of a solid object on a rough surface. In the absence of sliding we have the familiar nonholonomic problem. The solutions of this problem afford a good description of the actual state of affairs only if any small sliding motion due to causes not considered vanishes after a short time.

2. Let us consider a mechanical system subject to stationary holonomic ideal constraints with the holonomic coordinates $q_1, \ldots, q_{n+l+k}$, and with nonholonomic stationary $(\partial A_{ij} / \partial t = 0)$ ideal constraints

$$A_{s1}\dot{q}_1 + \ldots + A_{s,n+l+k}\, \dot{q}_{n+l+k} = 0 \qquad (s = 1, \ldots, k)$$

with the possible displacements defined by

$$A_{s1}\delta q_1 + \ldots + A_{s,n+l+k}\delta q_{n+l+k} = 0 \qquad (s = 1, \ldots, k)$$

Further, let the system be subject to the liberating constraints with dry friction

$$q_{n+1} \leqslant 0, \ldots, q_{n+l} \leqslant 0$$

If these inequalities become equations, the bodies or points of the system slide along the bodies of the system or external bodies; the frictional force is proportional to the normal reaction $N_i > 0$ and is directed opposite to the relative velocity of the sliding motion ($N_i > 0$ if the bodies exert pressure on one another). At each point of contact on one of the bodies we fix a triplet of axes with the z_i-axis directed along the exterior normal and the x_i- and y_i-axes rendering the triplet right-handed and rectangular. Then the work performed by the reaction N_i and the frictional force over a possible displacement δx_i, δy_i, δz_i of a point on the second body relative to this system is

$$N_i\delta z_i - k_iN_i \frac{v_{ix}}{|\mathbf{v}_i|}\, \delta x_i - k_iN_i \frac{v_{iy}}{|\mathbf{v}_i|}\, \delta y_i$$

where v_{ix}, v_{iy} are the projections of the relative sliding velocity $\mathbf{v}_i$ on the x_i- and y_i-axes and $k_i > 0$ is the coefficient of friction.

Let $q_{n+1} = \ldots = q_{n+l} = \dot{q}_{n+1} = \ldots = \dot{q}_{n+l} = 0$ be the initial conditions. We consider the complete system of nonholonomic variables

$$v_1, \ldots, v_n, v_{n+1} = \dot{q}_{n+1}, \ldots, v_{n+l} = \dot{q}_{n+l}$$

Let v_{ix}, v_{iy}, v_{iz} be the components of the possible relative velocity of a point on one of the bodies that are in contact relative to the system x_i, y_i, z_i; these components may be expressed in terms of $v_1, \ldots, v_{n+1}$ as

$$v_{ix} = \alpha_{i1}{}^1 v_1 + \ldots + \alpha_{i,n+l}^{\,1} v_{n+l}, \qquad v_{iy} = \alpha_{i1}{}^2 v_1 + \ldots + \alpha_{i,n+l}^{\,2} v_{n+l},$$
$$v_{iz} = \alpha_{i,n}^{\,3} v_{n+1} + \ldots + \alpha_{i,n+l}^{\,3} v_{n+l}$$

We note that the same formulas may be used to express the displacements δx_i, δy_i, δz_i in terms of the possible displacements $v_1\delta\tau, \ldots, v_{n+l}\delta\tau$.

At $t = 0$, let there be some zero velocities $v_{\nu+1,0} = \ldots = v_{p0} = 0$, among the initial relative sliding velocities $v_{10}, \ldots, v_{p0}$ all the rest of the latter being non-zero. Also, let R_{ix}, R_{iy} $(i = \nu + 1, \ldots, p)$ be the projections of the tangential reactions on the x_i - and y_i -axes. Denoting the energy of accelerations of the system by

$$S = \sum_{ij=1}^{n+l} a_{ij} v_i^{\cdot} v_j^{\cdot} + \sum_{j=1}^{n+l} b_j v_j^{\cdot} = S_2 + S_1$$

and the stationary and continuous generalized forces by $Q_1, \ldots, Q_{n+l}$ we find that the equations of motion become

$$\frac{\partial S_2}{\partial v_j^{\cdot}} = -b_j + Q_j + \sum_{i=1}^{\nu}\left(-k_i N_i \frac{\alpha_{ij}^1 v_{ix} + \alpha_{ij}^2 v_{iy}}{\sqrt{v_{ix}^2 + v_{iy}^2}} + \alpha_{ij}^3 N_i\right) + \tag{2.1}$$
$$+ \sum_{i=\nu+1} (\alpha_{ij}^1 R_{ix} + \alpha_{iy}^2 R_{iy} + \alpha_{ij}^3 N_i) = P_j$$

Denoting the right-hand sides of this system by P_j, we determine $q_j^{\cdot\cdot}$, $v_{jx}^{\cdot}$, $v_{jy}^{\cdot}$ from these equations and set the former equal to zero,

$$\begin{aligned} q_j^{\cdot\cdot} &= \gamma_{1j}P_1 + \ldots + \gamma_{n+l,j}P_{n+l} = 0 \qquad (j = n+1, \ldots, n+l) \\ v_{jx}^{\cdot} &= \delta_{1j}P_1 + \ldots + \delta_{n+l,j}P_{n+l} = 0 \qquad (i = \nu+1, \ldots, p) \\ v_{jy}^{\cdot} &= \theta_{1j}P_1 + \ldots + \theta_{n+l,j}P_{n+l} = 0 \qquad (j = \nu+1, \ldots, p) \end{aligned} \tag{2.2}$$

The resulting system of equations enables us to determine the normal reactions and frictional forces.

Now let $S^* = S$, where $q_j^{\cdot\cdot}$, $v_{jx}^{\cdot}$, $v_{jy}^{\cdot}$; are set equal to zero; let $Q_1^*, \ldots, Q_\sigma^*$ be the generalized forces that correspond to the independent nonholonomic variables $v_1', \ldots, v_\sigma'$ in a system subject to the additional ideal constraints $q_j^{\cdot} = v_{jx} = v_{jy} = 0$ (some of which may be nonholonomic).

Omitting the primes in the coefficients $\alpha_{ij}^{1\prime}$, $\alpha_{ij}^{2\prime}$ we write the equations

$$\frac{\partial S_2^*}{\partial v_j^{\cdot}} = b_j^* + Q_j^* + \sum_{i=1}^{\nu}\left(-k_i N_i \frac{\alpha_{ij}^1 v_{ix} + \alpha_{ij}^2 v_{iy}}{\sqrt{v_{ix}^2 + v_{iy}^2}}\right) \tag{2.3}$$

Let us also introduce some additional hypothesis about the properties of the reaction,

$$\varphi_1 (N_i, q_i, q_i^{\cdot}, Q_i) = \ldots = \varphi_r = 0 \tag{2.4}$$

An example of such a hypothesis is the supposition that a heavy homogeneous stool subjected to the action of forces applied in the rough plane surface of its support exerts equal normal pressures on that surface at all four of its points of support.

Let us assume that system (2.2) – (2.4) possesses the following properties:

2.1. All the normal pressures $N_1, \ldots, N_p$ can be set negative. This means that under the given initial conditions system (2.2), (2.4) does not imply that a normal reaction necessarily arises at some point of contact.

2.2. System (2.2), (2.4) enables one to determine unambiguously all the linear combinations of $N_1, \ldots, N_\nu$ that occur on the right-hand sides of equations (2.3).

2.3. On the basis of any system $N_{\nu+1}, \ldots, N_p$ it is possible to choose the tangential reactions R_{ix}, R_{iy} such that (2.2), (2.4) and the inequalities

$$k_i N_i > \sqrt{R_{ix}^2 + R_{iy}^2} + c \qquad (c = \text{const} > 0) \tag{2.5}$$

are jointly satisfied.

Inequalities (2.5) signify that the absolute value of the tangential reaction is less than that of the maximum possible reaction.

If conditions 2.1, 2.2, 2.3 are satisfied under the given initial conditions, subsequent motion occurs in accordance with equations (2.3) taken in conjunction with the equations of the nonholonomic constraints of the initial system and the equations

$$q_{n+1} = \ldots = q_{n+l} = 0, \qquad v_{jx} = v_{jy} = 0 \qquad (j = \nu + 1, \ldots, p)$$

2.4. Let us assume now that systems (2.2) – (2.4) which follow from the definition of motion also satisfy conditions 2.1, 2.2, 2.3 for any $x_{i0} = q_{i0}' - q_{i0}$, $x_{i0}^{\cdot} = q_{i0}'^{\cdot} - q_{i0}^{\cdot}$, sufficiently small in absolute value but not so small as to make all the $v_{\nu+1}, \ldots, v_p$, vanish. In addition, the reactions $N_{\nu+1}', \ldots, N_p'$, corresponding to these initial conditions satisfy the inequalities

$$k_i N_i' > \sqrt{R_{ix}^2 + R_{iy}^2} + c$$

where R_{ix}, R_{iy} are the reactions corresponding to the initial conditions as regards q_{i0}, $q_{i0}^{\cdot}$.

All of the foregoing limitations guarantee the existence of some region

$$\sum x_i^2 + \sum x_i^{\cdot 2} \leqslant H$$

such that if any of the sliding velocities $v_{\nu+1}, \ldots, v_p$ are set equal to zero at $t = 0$ in H, they will remain equal to zero at least until such time as the motion goes beyond the region H. If assumptions 2.1, ..., 2.4 are fulfilled, we say that the variables v_{ix}, v_{iy} ($i = \nu + 1, \ldots, p$), are at an interior point of the stagnation zone.

Let us consider a system of initial conditions under which none of the relative velocities $v_{\nu+1}, \ldots, v_p$ is equal to zero and take as our $v_{\sigma+1}', \ldots, v_n'$ some internally independent system chosen from among v_{ix}, v_{iy} ($i = \nu + 1, \ldots, p$). Computing $v_1'', \ldots, v_\sigma''$ in terms of $v_{\sigma+1}'', \ldots, v_n''$ from the first σ equations of system (2.3) and substituting these expressions into the last $n - \sigma$ equations, we obtain the following system:

$$\frac{\partial}{\partial v_j^{\cdot}} \frac{1}{2} \sum_{ij=\sigma+1}^{n} c_{ij} v_i^{\cdot} v_j^{\cdot} = \mu_j + \sum_{i=\nu+1}^{p} - \left(k_i N_i \frac{\alpha_{ij}^1 v_{ix} + \alpha_{ij}^2 v_{iy}}{\sqrt{v_{ix}^2 + v_{iy}^2}} \right) \tag{2.6}$$

where the sum in the left-hand side is a positive definite quadratic form relative to $v_j^{\cdot}$ ($j = \sigma + 1, \ldots, n$). This form is the end result of equating to zero the linear forms

$$\frac{\partial}{\partial v_j^{\cdot}} \left(\sum_{ij=1}^{n} a_{ij} v_i^{\cdot} v_j^{\cdot} \right) = 0 \qquad (j = 1, \ldots, \sigma) \tag{2.7}$$

in the quadratic portion of the energy of the system accelerations

$$S_2 = \sum_{ij=1}^{n} a_{ij} v_i^{\cdot} v_j^{\cdot}$$

For $j = \sigma + 1, \ldots, n$, the quantities v_{jx}, v_{jy} can be expressed solely in terms of $v_{\sigma+1}, \ldots, v_n$; this means that frictional forces at points with the subscripts $\nu + 1, \ldots, p$ do not appear in the first σ equations, and therefore that their right-hand sides depend continuously on their arguments in the neighborhood of

$$\mathbf{v}_{\nu+1} = \ldots = \mathbf{v}_p = 0$$

Thus, upon isolating μ_j (the constant term on the right-hand sides of the equations obtained by substitution), we can convert these equations to the form (2.6).

The tangential reactions R_{ix}, R_{iy} satisfy the equations

$$\mu_{i0} + \sum_{i=\nu+1}^{p} (\alpha_{ij}^1 R_{ix} + \alpha_{ij}^2 R_{iy}) = 0$$

for the initial values

$$q_{i0}, \dot{q}_{i0}, \mathbf{v}_{\nu+1,0} = \ldots = \mathbf{v}_{p0} = 0$$

Multiplying each equation of (2.6) by $v_j^{\cdot}$, adding, and taking into account (2.1) and (2.7), we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{ij=\sigma+1}^{n} c_{ij} v_i v_j\right) = \frac{1}{2}\sum_{ij=\sigma+1}^{n} \frac{dc_{ij}}{dt} v_i v_j + (\Delta\mu_j)\, v_j -$$

$$- \sum_{i=\nu+1}^{p} (k_i N_i \sqrt{v_{ix}^2 + v_{iy}^2} - R_{ix} v_{ix} - R_{iy} v_{iy})$$

Noting that the c_{ij} depend solely on the coordinates, we find that this equation may be represented as

$$\frac{d}{dt}\frac{1}{2}\sum^{n} c_{ij} v_i v_j = - \sum_{i=\nu+1}^{p} [k_i N_i \sqrt{v_{ix}^2 + v_{iy}^2} + (R_{ix} + \lambda_{ix})\, v_{ix} + (R_{iy} + \lambda_{iy})\, v_{iy}]$$

where all the λ_{ix}, λ_{iy} vanish if $x_i = 0$, $x_i^{\cdot} = 0$.

We now show that the region H can always be taken so small that the inequality

$$- \sum_{i=\nu+1}^{p} [k_i N_i \sqrt{v_{ix}^2 + v_{iy}^2} + (R_{ix} + \lambda_{ix})\, v_{ix} + (R_{iy} + \lambda_{iy})\, v_{iy}] < -\Theta \sqrt{T_\sigma}$$

$$\left(T_\sigma = \frac{1}{2}\sum_{ij=\sigma+1}^{n} c_{ij} v_i v_j\right)$$

is satisfied everywhere within it.

Let us denote by kN_{i_0} the lower bounds of the quantity kN_i and by λ_{ix}°, λ_{iy}° the upper bounds of $|\lambda_{ix}|$, $|\lambda_{iy}|$ in the region H. It is clear that no term on the left-hand side exceeds the expression

$$- [(k_i N_{i0} - \lambda_{ix}^\circ - \lambda_{iy}^\circ) \sqrt{v_{ix}^2 + v_{iy}^2} + R_{ix} v_{ix} + R_{iy} v_{iy}]$$

This expression is negative, provided that

$$(R_{ix}v_{ix} + R_{iy}v_{iy})^2 < (k_i N_{i0} - \lambda_{ix}^\circ - \lambda_{iy}^\circ)^2 (\sqrt{v_{ix}^2 + v_{iy}^2})^2$$

for which it is sufficient that the condition

$$R_{ix}^2 + R_{iy}^2 < (kN_{i0} - \lambda_{ix}^\circ - \lambda_{iy}^\circ)^2$$

be satisfied.

This condition is clearly satisfied if (2.4) holds true and the region H is sufficiently small.

Denoting by $-\gamma$ the negative maximum of the sum

$$\Phi = -\sum_{i=\nu+1}^{p} [(kN_i^\circ - \lambda_{ix}^\circ - \lambda_{iy}^\circ)\sqrt{v_{ix}^2 + v_{iy}^2} + R_{ix}v_{ix} + R_{iy}v_{iy}]$$

in the region H which is attained for $v_{\sigma+1}^*, \ldots, v_n^*$, and bearing in mind that Φ is homogeneous relative to velocities, we find that the inequality

$$-\Phi > \frac{\gamma\sqrt{v_{\sigma+1}^2 + \ldots + v_n^2}}{\sqrt{v_{\sigma+1}^{*2} + \ldots + v_n^{*2}}} > \Theta\sqrt{T_\sigma} \qquad (\Theta = \text{const} > 0)$$

is satisfied everywhere in the region H.

Integrating the inequality

$$dT_\sigma / dt < -\Theta\sqrt{T_\sigma}$$

we obtain

$$\sqrt{T_\sigma} - \sqrt{T_{\sigma 0}} < -{}^1/_2\Theta\,(t - t_0)$$

For any region H and any constant $\lambda < H$ we can find a number $t^*(\lambda H) > 0$, such that the time $t - t_0$ required for the system of initial conditions $\sum x_{i0}^2 + x_{i0}^{\cdot 2} \leqslant \lambda$ to leave the region exceeds $t^*(\lambda H)$. This implies that for all initial conditions for which

$$\sqrt{T_{\sigma 0}} - {}^1/_2\Theta t^*(\lambda, H) \leqslant 0$$

all $v_{\sigma+1}, \ldots, v_n$ vanish at the end of the time $t - t_0 \leqslant t^*$ and subsequent motion proceeds according to equations (2.3) until the system leaves the region H.

Let us formulate our final conclusions in the form of a theorem.

Theorem. (1) If the portion of the system corresponding to the variables $v_{\sigma+1}, \ldots, v_n$, is at an interior point of the stagnation zone under certain initial conditions, and if the region

$$\sum x_i^2 + x_i^{\cdot 2} \leqslant H, \qquad v_{\sigma+1} = \ldots = v_n = 0$$

consists entirely of interior points of this zone, then for any $\lambda < H$ it is possible to find a $\lambda_1(\lambda, H)$ such that for any initial conditions, motion from the region

$$\sum x_{i0}^2 + x_{i0}^{\cdot 2} \leqslant H, \qquad \sum_{i=\sigma+1}^{n} v_i^{\circ 2} \leqslant \lambda_1$$

until emergence from the region H occurs in two stages: in the first stage some of the relative sliding velocities are not equal to zero; in the second stage, which begins no later than a time $t - t_0 = 2\Theta^{-1}\sqrt{T_{\sigma 0}}$, all sliding velocities remain equal to zero until the motion leaves the region H.

(2) If the motion $q_i(t)$ beginning under the initial conditions q_{i0}, $q_{i0}^{\cdot}$, $v_{\sigma+1,0} = \ldots = v_{n0} = 0$, always remains at an interior point of the stagnation zone and is a stable solution of system (2.3) in conjunction with the equations $q_{n+1} = \ldots = q_{n-l} = v_{\sigma+1} = \ldots = v_n = 0$ (i.e. if the solution $q_i(t)$ is the stable motion of the initial system limited by the indicated additional constraints), then any disturbed motion of the initial system with sliding velocities sufficiently small in absolute value after a finite time interval becomes the motion of a system subject to additional constraints; the duration of the transient process tends to zero as $T_{\sigma 0} \to 0$.

BIBLIOGRAPHY

1. Painlevé, P., *Leçons sur le frottement* (*Lectures on friction*), Paris, 1897, Gostekhizdat, 1954 (in Russian).

2. Pozharitskii, G.K., Ob ustoichivosti ravnovesii dlia sistem s sukhim treniem (The stability of equilibria for systems with dry friction). *PMM*, Vol. 26, No. 1, 1962.

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